

TOE Dark Sector Extension: Gravitational/Computational Origin of Dark Matter and Dark Energy

Nick Rugai

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Research status and scope

This document is a downstream extension of the existing *Theory of Everything - All Is Epistemic* working monograph. It intentionally does **not** restate the TOE. Instead it assumes, conditionally and for the purpose of calculation, the existing TOE hypotheses: the 28-dimensional relational arena with 24 compact spatial dimensions, the candidate relational field I, the five-state spin-2 graviton qdit, operationally emergent time, and the requirement that four-dimensional general relativity and the Standard Model emerge as infrared limits.

The purpose here is deliberately stronger than asking whether the TOE can be made compatible with observations. The working instruction is:

Assume the TOE architecture is correct. Then force it to produce the dark sector.

The result is not presented as an experimentally confirmed explanation. Every TOE-specific step is a conjectural construction. Established cosmology and gravity are used as constraints, recovery targets, and falsification conditions.

The central strategy is to avoid adding independent dark-sector ontology unless the reduced TOE dynamics require it. In particular, this document does not introduce an arbitrary new dark-matter particle or an arbitrary dark-energy fluid. It asks whether the already-postulated compact gravitational sector and its collective vacuum/modulus degrees of freedom can generate the phenomena conventionally parameterized as dark matter and dark energy.

Executive result

The most economical conditional construction found here is a **two-regime dark sector generated by one parent gravitational substrate**:

1. **Dark matter candidate:** nonzero Kaluza-Klein eigenmodes of the compactified gravitational sector. After four-dimensional reduction, each stable massive spin-2 mode has five helicity states and therefore fits the TOE's five-state massive graviton/qdit interpretation. A sufficiently cold and long-lived population of these modes naturally redshifts approximately as a pressureless component, $\rho_{DM} \propto a^{-3}$, while interacting with ordinary matter primarily through gravity or suppressed wavefunction overlap.
2. **Dark energy candidate:** the homogeneous zero-mode of the compact-sector geometry together with the I field, after dimensional reduction, produces an effective four-dimensional potential V_{eff} . A stabilized or slowly rolling configuration can have pressure close to minus its energy density. The same construction can admit mild time evolution of the equation of state, which is useful because DESI DR2 analyses have reported evidence for evolving dark energy in some data combinations, while later July 2026 DESI Lyman-alpha full-shape results moved closer to Λ CDM and emphasized that the situation remains unsettled. [1,2]

The important conceptual point is that the two components do **not** require two unrelated substances. At the four-dimensional level they can be different macroscopic states of the same higher-dimensional gravitational/computational system:

$$\begin{array}{ll} \text{zero mode + compact vacuum/moduli} & \rightarrow \text{dark-energy-like sector ,} \\ \text{excited compact graviton modes} & \rightarrow \text{dark-matter-like sector .} \end{array}$$

The construction also exposes a serious limitation. A naive, pure exponential runaway of a 24-dimensional volume modulus does **not** automatically give $w \simeq -1$. For $n=24$, the canonical slope generated by the simplest volume scalings is order unity, which produces an equation of state far from a cosmological constant. Therefore the dark-energy problem is not solved by merely saying “extra dimensions.” The compact sector must have a stabilizing, plateau-like, near-stationary, or otherwise dynamically protected configuration. That failure mode is useful: it is a reality-veto target rather than something to hide.

1. The reduced problem

The previous TOE supplies a target microscopic arena. For the present purpose, only its gravitational/compact sector is needed. Let

$$M_{28} = M_{3,1} \times K_{24},$$

with full coordinates

$$X^A = (x^\mu, y^a), \mu = 0, 1, 2, 3, a = 4, \dots, 27.$$

Let the full metric be G_{AB} and let V_{24} denote the dimensionless or dimensionful volume of the compact manifold in a chosen coordinate convention. The 28D gravitational action has the schematic form already established in the TOE:

$$S_{28} = \int d^{28} X \sqrt{-G} \left[\frac{M_{28}^{26}}{2} (R_{28} - 2 \Lambda_{28}) + L_I + L_{matter} + L_q \right].$$

The immediate objective is to integrate out the compact coordinates without throwing away the very modes that could become dark matter. Thus the correct operation is not simply “compactify and forget.” It is

compactification $\rightarrow$ spectral decomposition $\rightarrow$ zero mode + massive tower $\rightarrow$ 4D effective action .

For a static reference compact metric $h_{ab}(y)$, define the scalar Laplacian spectrum by

$$-\nabla_K^2 Y_n(y) = \lambda_n Y_n(y), 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots.$$

A characteristic compact scale R_c allows the eigenvalues to be written as

$$\lambda_n = \frac{\hat{\lambda}_n}{R_c^2},$$

where $\hat{\lambda}_n$ is dimensionless. For an anisotropic 24-torus one may instead write

$$m_n^2 \simeq m_0^2 + \sum_{a=1}^{24} \frac{n_a^2}{R_a^2} + \Delta_{curv} + \Delta_{warp},$$

with $n = (n_1, \dots, n_{24})$. Exact shifts depend on the compact geometry and background fields.

3.1 Spectral density: the tower can be dense

For an approximately isotropic flat 24-torus with a common radius R_c and mode labels n in Z^{24} , the large-mode counting function obeys a Weyl-law estimate. Writing $N(m)$ for the number of modes with KK mass below m , one obtains parametrically

$$N(m) \sim \frac{\pi^{12}}{\Gamma(13)} (m R_c)^{24},$$

so that

$$\frac{dN}{dm} \sim \frac{24\pi^{12}}{\Gamma(13)} R_c^{24} m^{23}.$$

The precise coefficient depends on conventions and on the compact geometry, but the scaling is the important point: a 24-dimensional compact space can produce a very rapidly growing density of massive gravitational states. Therefore the phrase “the dark matter is the lightest KK mode” is not automatically self-consistent. If the spectrum is dense, many nearby modes can be populated, decay into one another, or contribute to cosmology. The TOE must solve the coupled tower problem or derive a dynamical selection rule that isolates a sparse subset.

The same density-of-states growth is also a quantum-consistency warning: summing vacuum energies or interaction rates over the tower without a UV completion can produce spurious divergences. The full 28D theory, not an arbitrarily truncated tower, is the object whose regulated observables must be used.

The four-dimensional Planck scale follows the usual dimensional-reduction structure,

$$M_{Pl}^2 = M_{28}^{26} V_{24},$$

for the reduced Planck-mass convention and a static compact volume. The precise normalization of V_{24} is conventional; the physical requirement is that the four-dimensional Newton constant be reproduced.

2. Why the graviton sector is the natural place to look

The previous TOE already gives the fundamental graviton a five-state spin-2 qdit interpretation:

$$\left| \psi_g \right\rangle = \sum_{m=-2}^2 a_m \left| 2, m \right\rangle, \quad \sum_{m=-2}^2 |a_m|^2 = 1.$$

The critical representation-theory fact is that a four-dimensional **massless** spin-2 field has two radiative helicities, while a four-dimensional **massive** spin-2 field has five physical polarization states. Therefore the same five-state object can have two very different infrared manifestations:

$$\text{massive regime: } m = -2, -1, 0, +1, +2,$$

versus

$$\text{massless radiative regime: } \lambda = \pm 2.$$

The TOE must still derive the precise projection mechanism. Here we use it as a target condition rather than as a completed theorem.

This makes compactification particularly interesting. Higher-dimensional gravitational fields naturally produce a massless four-dimensional zero mode plus massive Kaluza-Klein excitations. The existence of massive spin-2 states from extra-dimensional geometry is standard in the literature; consistent nonlinear massive-spin-2 theories are

technically delicate, and naive nonlinear Fierz-Pauli completions can propagate a Boulware-Deser ghost. Ghost-free massive gravity and higher-dimensional constructions therefore matter to the present program. [3,4]

The TOE advantage is that we are not inventing a massive spin-2 field by hand. We already have a higher-dimensional arena and a fundamental graviton sector. The massive states can be **spectral descendants** of the same gravitational substrate.

There is, however, a TOE-specific degree-of-freedom caveat that must stay visible. A generic higher-dimensional metric has more tensor components than the five physical polarizations of a four-dimensional massive spin-2 field. Thus the identification of the TOE five-state graviton qdit with a particular reduced KK tensor sector is an additional dynamical map, not something guaranteed by dimensionality alone. The full 28D constraint algebra, compactification, gauge fixing, and mode decomposition must show exactly which five states survive in the candidate massive sector.

3. Spectral reduction of the gravitational computational field

Write a perturbation of the higher-dimensional metric as

$$G_{AB} = \dot{G}_{AB} + H_{AB}.$$

The four-dimensional tensor component is expanded as

$$H_{\mu\nu}(x, y) = h_{\mu\nu}^{(0)}(x) Y_0(y) + \sum_{n>0} h_{\mu\nu}^{(n)}(x) Y_n(y)$$

with orthonormality

$$\int_{K_{24}} d^{24} y \sqrt{h} Y_n Y_m = \delta_{nm}.$$

The zero mode has $\lambda_0 = 0$. It is the mode identified with ordinary four-dimensional gravity:

$$S_0 \rightarrow \int d^4 x \sqrt{-g} \frac{M_{Pl}^2}{2} R[g] + \dots.$$

The nonzero modes acquire mass terms from internal gradients. At quadratic order their four-dimensional action has the Fierz-Pauli structure

$$S_{KK}^{(2)} = \sum_{n>0} \int d^4 x \sqrt{-g} \left[L_{kin}(h^{(n)}) - \frac{m_n^2}{2} (h_{\mu\nu}^{(n)} h^{(n)\mu\nu} - (h^{(n)})^2) \right].$$

The exact nonlinear completion is not optional. A candidate TOE must either retain the full higher-dimensional dynamics or derive a four-dimensional effective action whose massive-spin-2 sector is demonstrably free of the known pathologies over its domain of validity. The literature shows why this matters: generic nonlinear massive-spin-2 extensions can contain a sixth ghost degree of freedom, whereas special ghost-free constructions can retain five degrees of freedom. [3,4]

The matter coupling takes the generic form

$$S_{int} = -\frac{1}{2} \int d^4 x \sqrt{-g} \left[\frac{1}{M_{Pl}} h_{\mu\nu}^{(0)} T^{\mu\nu} + \sum_{n>0} \frac{C_n}{M_{Pl}} h_{\mu\nu}^{(n)} T^{\mu\nu} \right],$$

where C_n is an overlap/normalization factor determined by the compact geometry and by where the visible sector resides in the full 28D configuration. A dark mode is naturally dark when C_n is small or when an exact selection rule suppresses its decay into visible states.

For two point sources, the schematic static potential becomes

$$V(r) = -\frac{G m_1 m_2}{r} \left[1 + \sum_{n>0} c_n^2 e^{-m_n r} \Xi_n \right]$$

where Ξ_n encodes the spin/polarization structure. Thus the same modes that can store cosmological energy also leave a possible fifth-force/Yukawa signature if their Compton wavelengths are macroscopic. This creates an immediate experimental cross-check.

4. Dark matter from massive graviton qdits

4.1 The identification

The core conditional proposal is

dark matter $\equiv$ stable, nonrelativistic massive gravitational modes of the compact sector .

No new internal quantum number is required beyond the mode label and the spin-2 state. In the simplest version,

$$|n, m\rangle, n \neq 0, m = -2, \dots, +2.$$

The five-state qdit is therefore the polarization register of each massive mode, while n is its compactification mode index.

This proposal has a useful property: the phrase “dark matter particle” becomes a phenomenological description of a **known type of descendant of the TOE gravitational substrate**, rather than a new primitive.

4.2 Nonrelativistic limit

For a canonically normalized massive mode q_n , the local equation in an expanding background takes the leading form

$$\ddot{q}_n + 3H \dot{q}_n + \left(m_n^2 + \frac{k^2}{a^2} \right) q_n \simeq 0.$$

For a homogeneous coherent configuration with $m_n \gg H$,

$$q_n(t) \simeq a^{-3/2}(t) Q_n \cos(m_n t + \delta_n).$$

The averaged energy density is then of the form

$$\langle \rho_n \rangle \simeq \frac{1}{2} m_n^2 \langle q_n^2 \rangle \propto a^{-3},$$

while the oscillatory pressure averages toward zero:

$$\langle p_n \rangle \rightarrow 0.$$

This is the crucial dark-matter scaling. It is not assumed merely by naming the field dark matter; it emerges from the nonrelativistic massive-mode limit.

4.2.1 Five-polarization WKB reduction

To avoid importing scalar-field intuition too casually, decompose a canonically normalized massive spin-2 mode into its five physical polarizations,

$$h_{\mu\nu}^{(n)}(x) = \sum_{\lambda=-2}^2 q_{n\lambda}(x) e_{\mu\nu}^{(\lambda)}(k),$$

with the massive spin-2 constraints in vacuum

$$\partial^\mu h_{\mu\nu}^{(n)} = 0, h_\mu^{(n)\mu} = 0,$$

up to the corrections introduced by expansion, mixing, and interactions. In the local adiabatic/WKB limit each physical polarization has a leading oscillator frequency

$$\omega_{n,k}^2 \simeq m_n^2 + \frac{k^2}{a^2} + O(H^2, H m_n, \dot{H}).$$

The occupation-number representation then gives

$$E_{n,k} = \omega_{n,k}, n_n = a^{-3} \sum_{\lambda,k} f_{n\lambda}(k).$$

For $k/(a m_n) \ll 1$,

$$\omega_{n,k} = m_n + \frac{k^2}{2 m_n a^2} + O\left(\frac{k^4}{m_n^3 a^4}\right),$$

so the pressure contribution is parametrically

$$p_n \sim \frac{1}{3 a^2} \sum_{\lambda} \int \frac{d^3 k}{(2\pi)^3} \frac{k^2}{\omega_{n,k}} f_{n\lambda}(k),$$

while

$$\rho_n \simeq m_n n_n + \frac{1}{2 m_n a^2} \sum_{\lambda} \int \frac{d^3 k}{(2\pi)^3} k^2 f_{n\lambda}(k).$$

Hence $p_n/\rho_n \rightarrow 0$ as the physical momentum becomes small compared with m_n . The spin-2 nature survives in the five-state internal density matrix, but the leading background equation of state becomes dust-like. This is the precise sense in which a massive graviton qdit can behave as dark matter without requiring it to be a scalar.

A more general particle description gives

$$\rho_{KK} = \sum_n m_n n_n,$$

with

$$\dot{n}_n + 3 H n_n = -\Gamma_n n_n + \sum_{j>n} B_{j \rightarrow n} \Gamma_j n_j + C_n(T),$$

where C_n is a production collision term. The stable limit is $\Gamma_n \rightarrow 0$ for the cosmologically relevant lowest modes.

4.3 The qdit density matrix and isotropy

Because a spin-2 object carries internal polarization structure, a cosmological population cannot be treated only as a scalar number density unless the spin sector has been averaged appropriately. Let

$$\rho_g = \sum_{m,m'} \rho_{mm'} |2, m\rangle \langle 2, m'|.$$

For an isotropic unpolarized population,

$$\rho_g = \frac{1}{5} I_5,$$

and the spin entropy is

$$S_g = -k_B \text{Tr}(\rho_g \ln \rho_g) = k_B \ln 5.$$

A polarized state may produce anisotropic stress or direction-dependent signatures. Therefore the TOE must determine dynamically whether early-universe interactions drive the dark qdit ensemble toward an isotropic density matrix or instead leave a primordial tensor polarization.

This is a concrete advantage of the computational interpretation: the five-state internal state is not decorative. It becomes a physical bookkeeping device for the dark population.

5. Production of the dark graviton population

There are three broad TOE-compatible production channels.

5.1 Coherent production / misalignment

Suppose an early-universe transition leaves a nonzero classical amplitude $q_{n,prod}$ for a stable massive mode. Once

$$H(T_{prod}) \gtrsim m_n$$

is crossed into

$$H(T) \ll m_n,$$

the mode begins coherent oscillation. Its energy density at later times is approximately

$$\rho_n(T) = \rho_{n,prod} \left(\frac{a_{prod}}{a} \right)^3 = \rho_{n,prod} \left(\frac{T}{T_{prod}} \right)^3 \left(\frac{g_s(T)}{g_s(T_{prod})} \right),$$

up to the usual entropy-transfer factor.

The abundance condition is

$$\Omega_n h^2 = \frac{m_n s_0 Y_n}{\rho_{c,0}/h^2}$$

with

$$Y_n = \frac{n_n}{s}.$$

For a homogeneous coherent field, the equivalent initial-amplitude condition may be written schematically as

$$\rho_{n,prod} \simeq \frac{1}{2} m_n^2 q_{n,prod}^2.$$

Thus

$$q_{n,prod} \simeq \frac{\sqrt{2 \rho_{DM,prod}}}{m_n}.$$

This is useful because the abundance becomes an observable constraint on a microscopic TOE initial condition rather than an arbitrary relic-density parameter.

5.2 Gravitational freeze-in

If the mode is populated through extremely weak gravitational interactions, the asymptotic yield is obtained from the Boltzmann integral

$$Y_n(\infty) \simeq \int_{T_0}^{T_R} \frac{\langle \sigma v \rangle_n n_{eq}^2}{s H T} dT$$

for a production-dominated regime. The exact cross sections must be computed from the full compactified action. A four-dimensional contact estimate such as $\langle \sigma v \rangle \propto T^2/M_{Pl}^4$ is useful only as a scaling estimate; once T reaches the KK scale, the complete tower and higher-dimensional dynamics must be restored. Work on extra-dimensional spin-2 production explicitly shows that a truncated single-KK-mode EFT can fail at energies above the first graviton mass because cancellations among the full tower become important. [5]

That point is especially favorable to the TOE architecture: its fundamental description is already a full higher-dimensional computation rather than an isolated four-dimensional massive-spin-2 toy model.

5.3 Decay cascades within the tower

If higher modes are unstable but the lightest mode is protected by a discrete selection rule, the tower can evolve according to

$$\dot{\rho}_n + 3 H \rho_n = -\Gamma_n \rho_n + \sum_{j>n} B_{j \rightarrow n} \Gamma_j \rho_j.$$

For nonrelativistic decay daughters, total mass-energy remains approximately in matter-like form even as the mode composition migrates down the tower. The cosmological requirement is not necessarily that every KK level be stable. It is that the surviving component have a lifetime much longer than the relevant cosmological age and not inject excluded amounts of electromagnetic, hadronic, neutrino, or gravitational radiation.

6. Stability: the non-negotiable dark-matter condition

A dark-matter interpretation fails if the candidate disappears too quickly. The lifetime condition is

$$\Gamma_n^{-1} \gg t_{req},$$

where t_{req} is at least the age over which the observed abundance is required, typically comparable to the cosmic age for stable dark matter.

For an ordinary gravitationally coupled massive spin-2 particle, the dimensional decay scaling is roughly

$$\Gamma_n \sim C \frac{c_n^2 m_n^3}{M_{Pl}^2}$$

with a channel-dependent dimensionless coefficient C. This immediately shows why either a sufficiently small mass or a sufficiently small coupling/selection rule is needed.

The most attractive TOE-level stability mechanism is therefore not “it is dark because it is weakly coupled” alone. It is a **compact-sector selection rule**. Suppose an exact remnant discrete symmetry P_K exists with

$$P_K(h^{(0)}) = +1, P_K(h^{(1)}) = -1,$$

and all visible fields are even. Then the lightest odd gravitational mode cannot decay solely into even states:

$$h^{(1)} \rightarrow \nrightarrow \text{SM-only final state}.$$

The precise symmetry cannot simply be assumed; it must arise from a property of K_{24} and its boundary conditions. But this is a sharply defined derivation target.

7. The graviton-mass constraint and what it does to the proposal

The long-range gravitational-wave mode must remain extremely close to massless. The LIGO-Virgo-KAGRA GWTC-4.0 parameterized tests report a 90% credible upper bound of about

$$m_g \lesssim 1.92 \times 10^{-23} \text{ eV}/c^2$$

for the propagating graviton mass parameter used in their dispersion analysis, while finding no evidence for physics beyond GR in the tested waveforms. [6]

This does **not** automatically exclude the TOE dark-matter construction, because the construction distinguishes the ordinary gravitational zero mode from massive compact excitations:

$$h_{\mu\nu}^{(0)} \text{ versus } h_{\mu\nu}^{(n>0)}.$$

The constraint therefore becomes a design condition:

$$m_0 \rightarrow 0 \text{ in the observable 4D gravitational zero mode,}$$

while

$$m_{n>0} > 0 \text{ for candidate dark modes.}$$

The distinction is only legitimate if the compact reduction prevents dangerous mixing between $h_{\mu\nu}^{(0)}$ and $h_{\mu\nu}^{(n)}$. That mixing must be calculated, not verbally dismissed.

8. Structure formation: can five-state gravitons actually cluster?

The background scaling $\rho \propto a^{-3}$ is necessary but not sufficient. Dark matter must cluster and seed the observed matter distribution.

In the nonrelativistic regime, write each helicity component in a slowly varying envelope form

$$q_m(x, t) = \frac{1}{\sqrt{2m}} \left[\psi_m(x, t) e^{-imt} + \psi_m^{\dagger}(x, t) e^{+imt} \right].$$

The leading Schrödinger-Poisson system is then of the form

$$i \partial_t \psi_m = - \frac{\nabla^2}{2m a^2} \psi_m + m \Phi \psi_m,$$

and

$$\nabla^2 \Phi = 4 \pi G a^2 \left[m \sum_m |\psi_m|^2 + \delta \rho_b + \delta \rho_r + \dots \right].$$

For an ensemble with total number density n_g ,

$$\rho_g = m n_g.$$

The density perturbation obeys a generic competition between gravitational growth and wave/velocity pressure. In the simplest coherent nonrelativistic reduction, the effective sound-speed scaling is

$$c_s^2 \sim \frac{k^2}{4 m^2 a^2},$$

so the dark mode becomes indistinguishable from cold dust on scales where

$$\frac{k}{a} \ll m.$$

This supplies an immediate falsification program: if the lowest stable compact mode is too light, wave effects suppress structure below a calculable Jeans-like scale. If it is sufficiently heavy, the candidate approaches cold dark matter.

A useful characteristic scale is obtained by balancing the gravitational and wave terms,

$$k_J^4 \sim 16 \pi G \rho m^2 a^4,$$

up to order-unity factors depending on the precise spin-2 reduction. The TOE must ultimately derive those factors from its five-state tensor system rather than importing the scalar result unchanged.

9. Lensing and the metric seen by photons

A genuine dark-matter explanation must reproduce both dynamical mass and gravitational lensing. The observable lensing potential in Newtonian gauge is usually described by

$$d s^2 = - (1 + 2 \Psi) d t^2 + a^2(t) (1 - 2 \Phi) d x^2.$$

General relativity with minimally coupled cold matter gives, to leading order in the appropriate regime,

$$k^2 \Psi \simeq - 4 \pi G a^2 \rho \Delta,$$

and negligible anisotropic stress gives

$$\Phi \simeq \Psi.$$

A massive spin-2 dark population can in principle generate anisotropic stress. Therefore the TOE must calculate

$$\Pi_g \equiv \left(T_j^i \right)_{\text{traceless}}$$

and demonstrate that its cosmologically averaged contribution is consistent with observed lensing. An isotropic density matrix in qdit space is a natural candidate condition, but again it must be produced dynamically or established as an attractor.

The effective linearized lensing equation can be parameterized as

$$-k^2 (\Phi + \Psi) = 8 \pi G a^2 \Sigma(k, a) \rho_m \Delta_m.$$

The TOE must calculate $\Sigma(k, a)$, not fit it independently. In the successful GR-like regime,

$$\Sigma(k, a) \rightarrow 1.$$

A nontrivial prediction would be a controlled scale dependence caused by the first accessible compact mode.

10. Dark energy from the compact-sector vacuum

We now turn to the acceleration problem. The starting point is the same compact geometry; no separate “dark-energy particle” is introduced.

Let the internal metric contain a breathing mode $b(x)$,

$$h_{ab}(x, y) = b^2(x) \tilde{h}_{ab}(y),$$

with 24 internal dimensions. Define

$$\sigma = \ln b.$$

After the Weyl rescaling required to put the four-dimensional Einstein-Hilbert term in canonical form, the volume modulus has a canonical scalar normalization of the form

$$\phi = M_{Pl} \sqrt{\frac{n(n+2)}{2}} \sigma, n=24.$$

Thus

$$\phi = M_{Pl} \sqrt{312} \ln b \approx 17.66 M_{Pl} \ln b.$$

The reduced four-dimensional action can be organized as

$$S_4 = \int d^4x \sqrt{-g} \left[\frac{M_{Pl}^2}{2} R - \frac{1}{2} (\partial \phi)^2 - V_{eff}(\phi, I) + \sum_n L_K^{(n)} + L_{SM} \right].$$

The effective potential is a sum of contributions generated by the parent compact sector,

$$V_{eff} = V_{\Lambda_{28}} + V_{curv} + V_I + V_q + V_{int} + \dots.$$

For the simplest breathing-mode scalings,

$$V_{\Lambda_{28}} \propto e^{-n\sigma},$$

while internal curvature contributes

$$V_{curv} \propto e^{-(n+2)\sigma}.$$

Quantum compactification effects, boundary conditions, or other TOE fields may contribute additional powers or non-power-law structure.

11. The first reality veto: naive 24D volume runaway

Suppose, temporarily, that the potential were a single exponential

$$V(\phi) = V_0 e^{-\lambda \phi / M_{Pl}}.$$

Using

$$\phi = M_{Pl} \sqrt{\frac{n(n+2)}{2}} \sigma,$$

the exponential $e^{-\alpha\sigma}$ corresponds to

$$\lambda = \frac{\alpha}{\sqrt{n(n+2)/2}}.$$

For $n=24$:

$$\lambda_{\Lambda} = \frac{24}{\sqrt{312}} \approx 1.36,$$

and

$$\lambda_{curv} = \frac{26}{\sqrt{312}} \approx 1.47.$$

For a scalar-dominated exponential attractor,

$$w_{\phi} = -1 + \frac{\lambda^2}{3}.$$

Therefore the two simplest pure runaways would give approximately

$$w_{\Lambda} \approx -0.38,$$

and

$$w_{curv} \approx -0.28.$$

That is nowhere near a cosmological-constant-like state.

This is a valuable failure. It means the TOE cannot claim that dark energy is simply “the volume of the extra dimensions.” The basic 24D scaling is too steep in the canonical variable.

The dark-energy solution therefore requires one of the following to emerge from the fundamental theory:

- a stabilized minimum,
- an extended near-stationary plateau,
- a cancellation producing a locally shallow effective potential,
- a coupled multi-field attractor,
- a noncanonical kinetic structure,
- or a different emergent dark-energy stress tensor not reducible to a minimally coupled scalar.

We should prefer whichever option is *derived* rather than inserted.

12. Near-stationary compact-sector energy as dark energy

Assume the full compact-sector potential has a dynamically selected configuration near (ϕ_c, I_c) , with

$$\left. \frac{\partial V_{eff}}{\partial \phi} \right|_c = 0, \left. \frac{\partial V_{eff}}{\partial I} \right|_c = 0.$$

Then

$$\rho_{DE} = V_{eff}(\phi_c, I_c),$$

and

$$p_{DE} = -V_{eff}(\phi_c, I_c).$$

Hence

$$w_{DE} = -1$$

in the exactly frozen limit.

The magnitude condition is

$$V_{eff}(\phi_c, I_c) = \rho_{DE,0}.$$

Numerically, the observed dark-energy scale is of order a few millielectronvolts in the fourth root,

$$\rho_{DE,0}^{1/4} \sim 10^{-3} eV,$$

so the effective potential is extraordinarily small compared with high-energy particle scales. This familiar hierarchy is not solved merely by dimensional reduction. The TOE must either predict the small residual value, explain a technically natural cancellation, or provide a dynamical selection principle that makes the small value an attractor.

The equation of state itself gives a direct microscopic velocity target. From

$$1 + w_\phi = \frac{\dot{\phi}^2}{\rho_\phi},$$

one obtains today

$$\frac{|\dot{\phi}_0|}{H_0 M_{Pl}} \simeq \sqrt{3 \Omega_{\phi,0} (1 + w_0)}.$$

Thus a measured value of w_0 arbitrarily close to -1 forces the compact/I sector to be extremely slowly evolving in Planck units. This is not merely a fit parameter: it is a boundary condition on the microscopic rate at which the compact relational state may change.

For a multi-field compact sector with field-space metric $G_{AB}(\varphi)$, the corresponding invariant condition is

$$1 + w_{dark} \simeq \frac{G_{AB} \dot{\varphi}^A \dot{\varphi}^B}{\rho_{dark}},$$

so the same conclusion applies without privileging a particular scalar coordinate.

13. A dynamical dark-energy branch

The more interesting possibility is that the compact/I sector is not exactly frozen but evolving slowly. For canonical fields,

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V,$$

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V,$$

and therefore

$$w_\phi = \frac{\dot{\phi}^2/2 - V}{\dot{\phi}^2/2 + V}.$$

Equivalently,

$$1 + w_\phi = \frac{\dot{\phi}^2}{\rho_\phi}.$$

The Klein-Gordon equation is

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0.$$

The Friedmann equation becomes

$$3M_{Pl}^2 H^2 = \rho_r + \rho_b + \rho_{KK} + \rho_\phi.$$

The acceleration equation is

$$-2M_{Pl}^2 \dot{H} = \frac{4}{3}(\rho_r + \rho_b + \rho_{KK} + \dot{\phi}^2).$$

This branch is particularly relevant to current data because DESI DR2 has reported that a time-varying w_0, w_a model can improve fits to combined BAO/CMB/SN datasets, while later 2026 Lyman-alpha full-shape results shifted some measurements toward the standard Λ CDM prediction and emphasized that the evidence remains unsettled. [1,2]

The TOE therefore should not hard-code $w = -1$. It should derive $w(a)$ and let the data determine whether the frozen or rolling solution survives.

14. Direct mapping to the CPL dark-energy test

A common phenomenological expansion is

$$w(a) = w_0 + w_a(1 - a).$$

Energy conservation gives

$$\frac{d\rho_{DE}}{d \ln a} = -3[1 + w(a)]\rho_{DE},$$

and integration yields

$$\rho_{DE}(a) = \rho_{DE,0} a^{-3(1+w_0+w_a)} \exp[-3w_a(1-a)].$$

The TOE does not need to take w_0 and w_a as fundamental. They are derived observables. Given a microscopic potential V_{eff} , numerical integration produces $H(a)$, $\phi(a)$, and hence

$$w_0 = w(a=1),$$

and

$$w_a = - \left. \frac{dw}{da} \right|_{a=1}.$$

For a canonical scalar it is useful to define

$$\lambda(\phi) = -M_{Pl} \frac{V_{,\phi}}{V}.$$

Then the scalar dynamics can be represented by the standard autonomous variables

$$x = \frac{\dot{\phi}}{\sqrt{6} H M_{Pl}}, y = \frac{\sqrt{V}}{\sqrt{3} H M_{Pl}},$$

with

$$\Omega_\phi = x^2 + y^2, w_\phi = \frac{x^2 - y^2}{x^2 + y^2}.$$

The resulting phase-space equations are

$$\begin{aligned} \frac{dx}{dN} &= -3x + \sqrt{\frac{3}{2}} \lambda y^2 + \frac{3}{2} x [2x^2 + \gamma_b (1 - x^2 - y^2)], \\ \frac{dy}{dN} &= -\sqrt{\frac{3}{2}} \lambda x y + \frac{3}{2} y [2x^2 + \gamma_b (1 - x^2 - y^2)], \end{aligned}$$

where $N = \ln a$ and $\gamma_b = 1 + w_b$ for the dominant background fluid.

This gives an explicit computational bridge:

compactification geometry $\rightarrow V_{eff}(\phi, I) \rightarrow \lambda(\phi) \rightarrow H(a) \rightarrow w_0, w_a \rightarrow \text{DESI/BAO/SN/CMB likelihood}.$

15. Vacuum energy and Casimir structure

A second possibility is that the residual dark-energy density arises from the quantized compact gravitational sector rather than from a classical rolling modulus. Formally,

$$V_{vac}(K) = \frac{1}{2} \sum_j (-1)^{F_j} g_j \int \frac{d^3 k}{(2\pi)^3} \sqrt{k^2 + m_j^2(K)}.$$

Here j runs over all physical modes, including the graviton tower and TOE compact-sector fields, F_j distinguishes bosonic and fermionic statistics where applicable, and g_j counts physical states.

A regulator and renormalization prescription are mandatory. A heat-kernel representation is cleaner:

$$V_q(s) = \frac{\mu^{2s}}{2\Gamma(s)} \sum_j (-1)^{F_j} g_j \int_0^\infty dt t^{s-1} \frac{e^{-tm_j^2}}{(4\pi t)^{3/2}} K_j(t; K),$$

followed by analytic continuation to the renormalized value at $s=0$. The compact geometry changes K_j and the spectrum m_j , so quantum vacuum energy is in principle geometry dependent.

For a compactification mechanism to generate a stable dark-energy-like state, one needs simultaneously

$$V_{eff}'(\phi_c) = 0, V_{eff}''(\phi_c) \lesssim H_0^2, V_{eff}(\phi_c) \simeq \rho_{DE,0}.$$

The first equation gives stationarity; the second permits slow cosmological response; the third fixes the observed magnitude. If the system has no dynamical reason to satisfy these relations, the construction remains a parameterized model rather than an explanation.

16. Unified dark-sector bookkeeping

The reduced TOE can be organized into a single stress-energy budget:

$$T_{\mu\nu}^{eff} = T_{\mu\nu}^{SM} + T_{\mu\nu}^{rad} + T_{\mu\nu}^{KK} + T_{\mu\nu}^{comp/vac} + T_{\mu\nu}^{corr}.$$

The ordinary observer labels the last two groups phenomenologically. In the simplest successful branch,

$$T_{\mu\nu}^{KK} \approx \rho_{DM} u_\mu u_\nu,$$

and

$$T_{\mu\nu}^{comp/vac} \approx -\rho_{DE} g_{\mu\nu}.$$

The dark-sector total is therefore

$$\rho_{dark} = \rho_{KK} + \rho_{DE}, p_{dark} = p_{KK} + p_{DE}.$$

No independent dark-sector force is required at the background level.

A unified energy-transfer equation can be written

$$\dot{\rho}_{KK} + 3H \rho_{KK} = Q,$$

$$\dot{\rho}_{DE} + 3H(1+w_{DE})\rho_{DE} = -Q.$$

The simplest branch is $Q=0$. A more interesting branch has Q generated by the dependence of the KK masses on a compact modulus,

$$m_n = m_n(\phi),$$

so that

$$\frac{\dot{m}_n}{m_n} = \alpha_n \frac{\dot{\phi}}{M_{Pl}},$$

and energy exchange appears even if the underlying 28D theory has only one covariant stress tensor. The matter-sector continuity equation then follows from diffeomorphism invariance and the scalar coupling.

Such a coupling could address a coincidence-like relation between matter and dark-energy densities, but it can also destroy the successful dust scaling or produce fifth forces. Therefore Q is an optional hypothesis with a severe empirical burden, not a free improvement knob.

17. The coincidence problem inside the TOE

The present universe has both matter and dark-energy components of comparable cosmological relevance even though their uncoupled densities scale differently:

$$\rho_{DM} \propto a^{-3}, \rho_{DE} \approx \text{constant}$$

in the LambdaCDM limit. The equality epoch is therefore transient.

The unified compact-sector picture offers a possible structural link because both components are determined by the same spectral geometry:

$$\{R_a, \tilde{h}_{ab}, V_{eff}, \lambda_n, c_n\} \rightarrow \{m_n, \Omega_n, w(a)\}.$$

In other words, the coincidence is no longer two completely independent inputs. However, correlation is not explanation. The theory still has to show that the same compact geometry naturally produces the observed abundance without unacceptable tuning.

18. Anisotropic compactification and the dark hierarchy

An isotropic compactification is not required. Let the 24 radii be R_a . Then

$$m_n^2 = \sum_{a=1}^{24} \frac{n_a^2}{R_a^2} + \dots$$

This produces a dense spectrum if many R_a are comparable and a hierarchical spectrum if the radii are hierarchical.

The compact geometry therefore supplies a natural spectral design space:

$$R_a \uparrow \Rightarrow m_n \downarrow,$$

while

$$R_a \downarrow \Rightarrow m_n \uparrow.$$

But every low mass introduces a longer-range fifth-force and cosmological constraint. The correct procedure is an inverse spectral problem:

$$\text{find admissible } K_{24} \text{ such that } \left(\begin{array}{l} M_{Pl} \text{ is recovered,} \\ \text{SM/gauge sector is recovered,} \\ \text{one stable massive mode gives } \Omega_{DM}, \\ \text{vacuum/modulus sector gives } \Omega_{DE}, \\ \text{gravity tests pass,} \\ \text{structure/lensing pass,} \\ \text{no ghost/anomaly instability occurs.} \end{array} \right)$$

This is a well-defined research program rather than a slogan.

19. The dark-energy option that naturally survives evolving-w tests

The most useful target is not an arbitrary phantom fluid. It is a slowly moving effective potential with

$$-1 \lesssim w(a) < -\frac{1}{3}$$

for the acceleration epoch, while approaching an approximately matter/radiation-dominated cosmology at earlier times.

A sufficient slow-roll-like hierarchy is

$$\epsilon_V = \frac{M_{Pl}^2}{2} \left(\frac{V_{,\phi}}{V} \right)^2 \ll 1,$$

and

$$|\eta_V| = \left| M_{Pl}^2 \frac{V_{,\phi\phi}}{V} \right| \lesssim 1.$$

To keep the field evolving on a Hubble timescale rather than rapidly, the effective mass should be roughly

$$m_\phi^2 = V_{,\phi\phi} \sim H_0^2$$

in a thawing branch, or substantially below H_0^2 in an initially frozen branch.

The 24D volume normalization makes a generic single exponential too steep, as shown above. Hence the TOE must generate a **local flattening**. One possible route is cancellation among compact-sector contributions:

$$V_{eff} = A e^{-24\sigma} + B e^{-26\sigma} + C e^{-\beta\sigma} + V_I(\sigma, I).$$

At a stationary point,

$$V_{,\sigma} = -24 A e^{-24\sigma} - 26 B e^{-26\sigma} - \beta C e^{-\beta\sigma} + V_{I,\sigma} = 0.$$

A second condition can make the curvature small,

$$V_{,\sigma\sigma} = 576 A e^{-24\sigma} + 676 B e^{-26\sigma} + \beta^2 C e^{-\beta\sigma} + V_{I,\sigma\sigma} \approx 0.$$

If such relations emerge from symmetry or geometry, the resulting flat direction can be technically meaningful. If they are inserted by hand through unrelated constants, the model merely relocates fine-tuning.

20. The role of the I field

The existing TOE already contains the candidate relational I field. It should therefore be tested as a dark-energy degree of freedom before inventing another scalar.

Use the four-dimensional effective action

$$S_I = \int d^4x \sqrt{-g} \left[-\frac{1}{2} Z(I, \phi) (\partial I)^2 - V_I(I, \phi) \right].$$

The canonical field exists locally whenever $Z > 0$. The null-energy condition requires

$$\rho_I + p_I = Z \dot{I}^2 \geq 0,$$

so a healthy canonical branch cannot cross into arbitrary phantom behavior merely to fit data.

The coupled equations are

$$\ddot{I} + 3H\dot{I} + \frac{1}{Z} V_{,I} + \text{kinetic-metric corrections} = 0,$$

and

$$3M_{Pl}^2 H^2 = \rho_r + \rho_b + \rho_{KK} + \rho_\phi + \rho_I.$$

The computational interpretation suggests one additional condition: the I field should represent a physical relational degree of freedom, not a symbol introduced solely because cosmology needs a scalar. Its equation, coupling, and normalization must be derived from the same 28D parent theory.

21. A possible graviton-vacuum equation of state

There is another branch worth preserving as a formal possibility: the quantum gravitational sector itself could generate a stress tensor that is approximately proportional to the metric,

$$\langle T_{\mu\nu}^{g,vac} \rangle = -\rho_g^{vac} g_{\mu\nu}.$$

By definition this gives

$$p_g^{vac} = -\rho_g^{vac}, w_g^{vac} = -1.$$

The nontrivial problem is not the equation of state. Lorentz invariance of a true vacuum already forces this tensor form. The hard problem is its **renormalized magnitude**:

$$\rho_g^{vac} \stackrel{?}{\approx} \rho_{DE,0}.$$

The TOE therefore cannot declare victory simply because the tensor has the correct form. It must calculate the finite residual after the full spectrum, compactification, renormalization, and backreaction are treated consistently.

22. Backreaction and self-consistency

The compact geometry cannot be solved independently of the dark population. The full effective equations have the form

$$G_{\mu\nu} = \frac{1}{M_{Pl}^2} \left(T_{\mu\nu}^{SM} + T_{\mu\nu}^{KK} + T_{\mu\nu}^I + T_{\mu\nu}^{vac} + T_{\mu\nu}^{geom} \right).$$

The geometry determines the spectrum,

$$K_{24} \rightarrow \{m_n, c_n\},$$

but the populated spectrum contributes back into the geometry,

$$\{m_n, c_n, n_n\} \rightarrow T_{\mu\nu}^{KK} \rightarrow g_{\mu\nu}, K_{24}.$$

Therefore the real dark-sector solution is a coupled fixed-point problem:

$$K_{24}^{(\dot{t})} = F \left[\rho_{KK} \left(K_{24}^{(\dot{t})} \right), V_{eff} \left(K_{24}^{(\dot{t})} \right), T_{SM} \right].$$

This feedback loop is likely where the TOE can generate genuinely novel predictions. A geometry selected without matter backreaction is only a first approximation.

23. A compact derivation of the late-time cosmology

Assume the successful branch has one stable massive mode $n=1$ plus one slowly evolving compact/I field ϕ . The late-time system is

$$3 M_{Pl}^2 H^2 = \rho_r + \rho_b + \rho_g + \frac{1}{2} \dot{\phi}^2 + V(\phi),$$

$$\dot{\rho}_g + 3 H \rho_g = 0,$$

$$\ddot{\phi} + 3 H \dot{\phi} + V_{,\phi} = 0.$$

Then

$$\rho_g(a) = \rho_{g,0} a^{-3},$$

and

$$\rho_\phi(a) = \frac{1}{2} \dot{\phi}^2 + V.$$

The acceleration condition is

$$\frac{\ddot{a}}{a} = - \frac{1}{6 M_{Pl}^2} [\rho_r + \rho_b + \rho_g + \rho_\phi + 3 p_r + 3 p_\phi] > 0.$$

Since $p_r = \rho_r/3$ and p_g approximately zero,

$$\frac{\ddot{a}}{a} > 0 \Leftrightarrow \rho_\phi + 3 p_\phi < -(\rho_r + \rho_b + \rho_g) - 3 p_r.$$

In the field-dominated slow-roll limit,

$$\dot{\phi}^2 \ll V,$$

so

$$\rho_\phi + 3 p_\phi \approx -2 V,$$

and acceleration follows once V dominates the matter/radiation terms.

24. A stronger unified proposal: spectral dark sector

The most conceptually coherent version of the construction is therefore not “dark matter plus dark energy” as two added terms. It is a **spectral decomposition of the compact gravitational computation**:

$$G_{28} \xrightarrow{K_{24}} \left[G_{\mu\nu}^{(0)}, \{ h_{\mu\nu}^{[n]} \}_{n>0}, \phi, I, \text{vacuum functional} \right].$$

The observable labels then emerge:

$$G^{(0)} \rightarrow \text{ordinary gravity},$$

$$\{ h^{[n>0]} \}_{stable, NR} \rightarrow \text{dark matter},$$

$$\{ \phi, I, \text{vacuum} \}_{homogeneous} \rightarrow \text{dark energy}.$$

The dark sector is thus a statement about **which sector of the same computation is populated and at what scale**.

25. Required observable outputs

The extension is scientifically useful only if it outputs a finite set of measurable quantities. At minimum:

$$O = \{ M_{Pl}, H(a), \Omega_{KK}, m_1, \tau_1, c_1, P(k), f\sigma_8, \Sigma(k, a), \Phi - \Psi, w_0, w_a, \text{GW dispersion} \}.$$

The TOE-specific calculation should therefore end in a likelihood comparison of the form

$$L_{total} = L_{CMB} L_{BAO} L_{SN} L_{lens} L_{LSS} L_{GW} L_{localgravity} L_{lab}.$$

The point is not to maximize a single score and declare a winner. The point is to expose which independent measurements the theory satisfies and where it fails.

26. Parameter economy and the “no new knobs” rule

The dark-sector extension should inherit parameters from the parent TOE rather than create a new cosmological parameter zoo.

A useful parameter ledger is:

Parameter	Origin	Observable role
M_{28}	28D gravitational normalization	fixes 4D Planck scale with compact volume
R_a	compact geometry	KK masses and corrections to gravity
shape moduli	K_{24}	degeneracies and mode overlaps
$\hat{\lambda}_n$	compact Laplacian spectrum	KK mass ratios
c_n	overlap/selection structure	decay and fifth-force strength
V_I	existing I sector	dark-energy dynamics
compact curvature	K_{24} geometry	part of V_{eff}
quantum determinant	full spectrum	Casimir/vacuum contribution
initial qdit occupations	early-universe state	relic abundance

The proposed rule is:

No dark-sector parameter may be introduced if it can be derived from an existing TOE object.

Only genuinely irreducible quantities may remain free, and every such quantity must be reported explicitly.

27. Failure modes we should actively try to produce

A serious extension should contain its own kill switches.

Failure A: no stable KK mode

If every massive mode decays too quickly,

$$\Gamma_n \gg H_0,$$

then the KK-dark-matter branch dies.

Failure B: excessive fifth force

If some low mode gives

$$\sum_n c_n^2 e^{-m_n r}$$

large enough to conflict with laboratory, Solar System, binary-pulsar, or cosmological tests, that compact geometry dies.

Failure C: wrong structure growth

If the transfer function

$$T_{TOE}(k)$$

suppresses small-scale structure outside observational bounds, that mode spectrum dies.

Failure D: anisotropic stress

If

$$|\Phi - \Psi|$$

is too large, the spin-2 dark population dies as a dominant cosmological component unless a compensating mechanism is derived.

Failure E: dark-energy potential too steep

If

$$|V_{,\phi}|/V$$

is generically order $1/M_{Pl}$ and no attractor/plateau exists, the volume-modulus dark-energy branch dies.

Failure F: wrong dark-energy evolution

If the predicted

$$(w_0, w_a)$$

region is excluded by the combined cosmological data, the particular compact/I potential dies even if the broader TOE survives.

Failure G: unacceptable early dark energy

If

$$\Omega_{DE}(z)$$

is too large at recombination, during structure formation, or at other constrained epochs, the branch dies.

Failure H: gravitational-wave deviation

If KK mixing modifies the tensor propagation law enough to violate GW speed, dispersion, polarization, or damping bounds, the branch dies.

28. The most important mathematical inversion

The actual research problem can now be stated as a constrained inverse problem.

Given observational targets O_{obs} , find compact-sector data θ_K such that

$$\theta_K^i = \arg \min_{\theta_K} D \left[O_{TOE}(\theta_K), O_{obs} \right]$$

subject to

$$C_{ghost} = 0,$$

$$C_{anomaly} = 0,$$

$$C_{GR} = 0 \text{ within experimental precision,}$$

$$C_{SM} = 0 \text{ for the recovered low-energy particle spectrum,}$$

and

$$C_{stability} = 1.$$

Here D is not a sacred scalar score; it is an explicit collection of independent discrepancies. The epistemic rule is to preserve the discrepancy vector so that one failure is not hidden by ten successes.

29. A concrete numerical program

The next stage can be computationally explicit.

Stage 1: spectral geometry. Choose a controlled family of compact 24D manifolds, initially beginning with diagonal tori and simple homogeneous spaces. Compute

$$\{ \lambda_n, Y_n, c_n \}_{n=0}^N$$

at high N.

Stage 2: dimensional reduction. Derive the quadratic and cubic four-dimensional action, including all scalar, vector, and tensor mixing. Verify that the massive spin-2 sectors contain five and only five physical degrees of freedom in the regime used.

Stage 3: abundance. Solve the full Boltzmann system for the KK tower,

$$\dot{n} + 3Hn = C(n, T) - \Gamma n.$$

Compute $\Omega_{KK} h^2$.

Stage 4: linear cosmology. Insert the surviving modes into a Boltzmann/EFT cosmology solver. Compute the CMB, BAO, matter power spectrum, lensing, and growth observables.

Stage 5: compact-sector vacuum. Compute V_{eff} including classical curvature, the I field, and quantum determinants. Search for stable or metastable solutions satisfying

$$V_{,i} = 0, \text{ eig}(V_{,ij}) \geq 0$$

with at least one eigenvalue allowed to be very small if a dynamical DE branch is desired.

Stage 6: late-time likelihood. Integrate $H(a)$ and derive $w(a)$, w_0 , w_a . Compare to DESI, CMB, SN, lensing, and growth measurements without tuning to any one dataset in isolation.

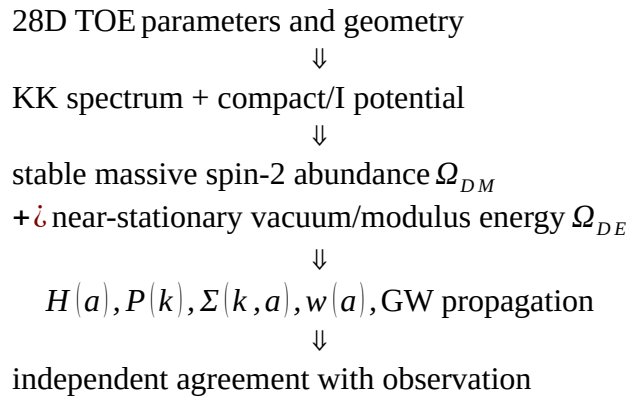
Stage 7: local constraints. Compute the Yukawa correction

$$\Delta V(r) = -\frac{G m_1 m_2}{r} \sum_{n>0} c_n^2 e^{-m_n r} \Xi_n$$

and compare to fifth-force, equivalence-principle, binary-pulsar, Solar System, and gravitational-wave constraints.

30. What would constitute a genuine success?

A successful dark-sector extension would not merely reproduce the observed numbers after tuning. It would achieve the following hierarchy:



The strongest possible version would additionally predict a nontrivial dark-sector signature, for example a specific mass spacing in the KK spectrum, a weak but calculable scale dependence in lensing or growth, a gravitational-wave propagation feature, or a specific deviation of $w(a)$ from -1.

The strongest null result would also be scientifically valuable. If the full calculation proves that no admissible K_{24} produces the observed dark sector while retaining the established low-energy world, the TOE has learned something decisive about its own architecture.

31. Final synthesis

The dark-sector problem looks different when entered from the existing TOE instead of bolted onto it afterward.

The central conditional construction is:

$$\text{same 28D graviton/computational substrate} \rightarrow \begin{cases} \text{excited compact modes} & \rightarrow \text{dark matter-like matter density} \\ \text{compact/I vacuum configuration} & \rightarrow \text{dark energy-like negative pressure} \end{cases}$$

Dark matter then has a natural microscopic interpretation as **massive spin-2 qdit excitations associated with compact structure**. Dark energy is not identified with a new substance at all; it is the effective stress-energy of a slowly varying or stabilized state of the same compact/I system.

The strongest result of this exercise is not that we have “solved” dark matter and dark energy. We have not. The useful result is that the existing TOE architecture can be pushed into a sharply defined set of mathematical obligations without importing arbitrary ontology.

And the architecture already provides an unexpectedly clean division:

zero-mode graviton $\leftrightarrow$ ordinary GR ,
massive compact graviton qdits $\leftrightarrow$ candidate dark matter ,
compact/I vacuum and slow modes $\leftrightarrow$ candidate dark energy .

The no-shortcut conclusion is equally important:

The compactification must generate the masses, couplings, stability, abundance, growth, lensing, and vacuum dynamics from the same underlying data. If those quantities have to be independently dialed, the TOE has not explained the dark sector; it has merely renamed it.

That is the reality-veto standard for this extension.

Appendix A. Compactification algebra for 24 internal dimensions

Take the Einstein-frame ansatz

$$ds_{28}^2 = e^{-n\sigma(x)} g_{\mu\nu}(x) dx^\mu dx^\nu + e^{2\sigma(x)} R_c^2 \tilde{h}_{ab}(y) dy^a dy^b, n=24.$$

The determinant scales as

$$\sqrt{-G} = e^{-n\sigma} \sqrt{-g} R_c^n \sqrt{\tilde{h}}.$$

The four-dimensional Einstein term emerges with constant coefficient because the Weyl factor compensates the changing internal volume. The scalar kinetic term takes the canonical form

$$-\frac{1}{2}(\partial\phi)^2,$$

with

$$\phi = M_{Pl} \sqrt{\frac{n(n+2)}{2}} \sigma.$$

An internal curvature scalar scales as

$$R_{int} = R_c^{-2} e^{-2\sigma} \tilde{R}_{int},$$

so its four-dimensional potential scales as

$$V_{curv}(\sigma) = \frac{1}{2} M_{Pl}^2 R_c^{-2} \tilde{R}_{int} e^{-(n+2)\sigma}.$$

The higher-dimensional cosmological term scales as

$$V_\Lambda(\sigma) = M_{Pl}^2 \Lambda_{28} e^{-n\sigma}.$$

These scaling relations are sufficient to expose the steep-exponential problem described in the main text.

Appendix B. Massive spin-2 consistency conditions

The quadratic Fierz-Pauli action is

$$L_{FP} = -\frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + \partial_\mu h^{\mu\nu} \partial^\lambda h_{\lambda\nu} - \partial_\mu h^{\mu\nu} \partial_\nu h + \frac{1}{2} \partial_\lambda h \partial^\lambda h - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2).$$

The field equations imply the constraints, schematically,

$$\partial^\mu h_{\mu\nu} = \partial_\nu h,$$

and

$$h = 0$$

in vacuum at the linear level, leaving five physical degrees of freedom.

At nonlinear order, generic extensions introduce an additional sixth propagating mode. The TOE should therefore obtain the massive modes as a true reduction of the complete higher-dimensional theory or derive a nonlinear interaction structure known to preserve the required constraint. This is not a technical footnote; it is a structural requirement. [3,4]

Appendix C. Dark-matter relic density equations

For a stable particle-like mode,

$$\frac{dn}{dt} + 3Hn = C[n, T],$$

and with $Y = n/s$,

$$\frac{dY}{dT} = -\frac{C[n, T]}{sHT}$$

when entropy in a comoving volume is conserved.

For decay-dominated evolution,

$$\frac{dn_n}{dt} + 3Hn_n = -\Gamma_n n_n + \sum_j B_{j \rightarrow n} \Gamma_j n_j.$$

The present density is

$$\rho_{n,0} = m_n n_{n,0} = m_n s_0 Y_{n,0}.$$

The abundance matching condition is

$$\Omega_n h^2 = \frac{m_n s_0 Y_{n,0}}{\rho_{c,0}/h^2}.$$

Appendix D. Dark-energy diagnostics

For any effective component,

$$w(a) = -1 - \frac{1}{3} \frac{d \ln \rho}{d \ln a}.$$

Therefore a measured or calculated density history immediately determines the equation of state.

For a scalar,

$$w(a)+1=\frac{\dot{\phi}^2}{\rho_{\phi}}.$$

The exact Lambda limit is therefore equivalent to

$$\dot{\phi}=0,$$

while an evolving branch has $\dot{\phi}$ nonzero.

For the CPL parameterization,

$$w(a)=w_0+w_a(1-a),$$

and

$$\rho(a)=\rho_0 a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)}.$$

Appendix E. Observational anchoring

The current empirical literature remains an important moving target. The 2025 DESI DR2 BAO cosmology analysis reported that a time-evolving dark-energy model could improve the fit to combinations of DESI BAO, CMB, and supernova data, while the July 2026 DESI DR2 Lyman-alpha full-shape release reported a measurement that shifted toward the fiducial Λ CDM prediction and explicitly noted that the evolving-dark-energy interpretation may weaken or require a more complex model. [1,2]

For the gravitational sector, the LVK GWTC-4.0 tests report no evidence for deviations from GR in the tested data and give a 90% credible graviton-mass upper bound of approximately 1.92×10^{-23} eV/c² in their dispersion analysis. [6]

These facts are not used as proof of the TOE. They define the empirical wall that any future calculation must hit.

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Research-status note

This extension is a speculative theoretical-physics research artifact. The mathematical equations from established GR, field theory, compactification, and cosmology are used as constraints and calculation templates. The identification of the TOE's compact gravitational modes with dark matter, and the identification of the compact/I vacuum sector with dark energy, are hypotheses to be derived, tested, and potentially rejected.

The central scientific rule remains:

One assumption. Everything else earns its place.